

FACTORIZATION OF DIFFERENTIABLE MAPS WITH BRANCH SET DIMENSION AT MOST $n - 3$ ⁽¹⁾

BY
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1. Introduction. Let M^n and N^n be separable n -manifolds (without boundary, unless otherwise specified). The *branch set* B_f of a map $f: M^n \rightarrow N^n$ is the set of points in M^n at which f fails to be a local homeomorphism. The map f is *monotone* if, for each $y \in N^n$, $f^{-1}(y)$ is a continuum, and f is *proper* if, for each compact set $X \subset N^n$, $f^{-1}(X)$ is compact.

1.1. THEOREM. *Let M^n and N^n be connected C^m manifolds ($m \geq 3$), and let $f: M^n \rightarrow N^n$ be C^m and proper with $\dim(B_f) \leq n - 3$. Then there is a factorization $f = hg$ such that*

- (1) $g: M^n \rightarrow K^n$ is a C^m monotone map onto the C^m n -manifold K^n ; and
- (2) $h: K^n \rightarrow N^n$ is a k -to-1 C^m diffeo-covering map. Moreover,
- (3) if $\bar{h}\bar{g}$ is another such factorization with intermediate space L^n , then there is a C^m diffeomorphism α of K^n onto L^n such that $\bar{g} = \alpha g$ and $\bar{h} = h\alpha^{-1}$.

The differentiability condition C^m may be replaced by C^∞ or real analytic. If M^n and N^n are compact oriented manifolds, then k is the absolute value of the degree of f .

1.2. COROLLARY. *Let M^n and N^n be compact connected oriented C^3 manifolds, and let $f: M^n \rightarrow N^n$ be C^3 . If N^n is simply connected and degree $f \neq \pm 1$, then $\dim(B_f) \geq n - 2$.*

1.3. The outline of the proof of (1.1). In (2.1) it is shown that the existence of the desired factorization is equivalent to two topological properties ((1) and (2)), and the remainder of the paper is devoted to showing that if f satisfies the hypotheses of (1.1), then it satisfies these properties. In (2.4), conclusion (1) is proved in a very special case, and (2.4) is used in (2.8) to deduce property (1) in case $n \geq 4$ and $f(B_f) \subset f(R_{n-2}(f))$ (definition below). In (3.5) it is shown that, given any map f satisfying the hypotheses of (1.1) (with $n \geq 4$), there is another map h such that, for each $y \in N^n$, $h^{-1}(y)$ and $f^{-1}(y)$ have the same number of components and h satisfies the hypothesis of (2.8); it follows that f satisfies condition (1). That it satisfies condition (2) is shown in (3.2). The cases $n \leq 3$ are treated separately in (3.6), and the uniqueness in (1.1) is also given in (3.6).

The set of points $x \in M^n$ at which the Jacobian matrix of f has rank at most q is denoted by $R_q(f)$ or, if there is no ambiguity, by R_q . Coordi-

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nates are written up, e.g., x^i, f^i , and $f|X$ is the restriction of f to the set X . The interior of X is denoted by $\text{int } X$, the boundary by $\text{bdy } X$, and closure by $\text{Cl}[X]$ or \bar{X} . The distance between x and y is $d(x, y)$, and $S(x, \delta)$ is the sphere about x of radius δ ; a map is a continuous function. Čech homology and cohomology are consistently used.

2. The proof of (1.1) in case $f(B_\rho) \subset f(R_{n-2})$.

2.1. **LEMMA.** *Let M^n and N^n be connected C^m n -manifolds, let $f: M^n \rightarrow N^n$ be a proper C^m map, and let $k = 1, 2, \dots$. Suppose that, for each point $y \in N^n$,*

(1) $f^{-1}(y)$ has exactly k components, and

(2) for each region U and each component V of $f^{-1}(U)$, $f(V) = U$.

Then f has the factorization of Theorem (1.1).

For M^n compact, a map f satisfying (2) is *quasi-monotone* [24, pp. 151, 152, (8.1)].

Proof. The manifold $N^n = \bigcup_{i=1}^\infty X_i$, where X_i is compact and $X_i \subset X_{i+1}$ ($i = 1, 2, \dots$). Since f is proper, $f|f^{-1}(X_i)$ has a unique monotone-light factorization [24, p. 141]; it follows that f also is hg , where g is monotone and h is light [24, p. 130, (4.4)]. Given any point $y \in N^n$, there is a connected open set U containing y such that the k components of $f^{-1}(y)$ are in different components of $f^{-1}(U)$. By (1) and (2), $f^{-1}(U)$ has exactly k components. Thus $h^{-1}(U)$ also has k components [24, p. 138, (2.2)] and h is 1-to-1 on each. By the local compactness of M^n and the Theorem on Invariance of Domain, h is a local homeomorphism. Hence h is a k -to-1 covering map [18, p. 128]. A natural C^m structure is thus induced by h on $K^n = g(M^n)$, so that g is C^m and h is a C^m diffeo-covering map.

2.2. **LEMMA.** *Let $f: M^2 \rightarrow N^2$ be a nonconstant map, where M^2 and N^2 are connected compact 2-manifolds, possibly with boundary. Suppose that*

$$\dim(f(B_\rho)) \leq 0, \quad f(B_\rho) \cap \text{bdy}(N^2) = \emptyset, \quad f| [M^2 - f^{-1}(f(B_\rho))]]$$

is a k -to-1 covering map, and $H^1(f^{-1}(y); Z_2) = 0$ for each $y \in N^2$. Then there exists a unique factorization $f = hg$, where g is a monotone map of M^2 onto M^2 and h is a light open map of M^2 onto N^2 .

Proof. Since f is nonconstant and $\dim(f(B_\rho)) \leq 0$, $f(M^2)$ meets $N^2 - f(B_\rho)$; from the covering property of f , it follows that f is quasi-monotone [24, p. 152, (8.1)]. Let hg be the monotone-light open factorization [24, p. 153, (8.4)] of f ; it is unique [24, p. 141, (4.1)]. Since $h(g(M^2))$ is both open in N^2 and compact, and since N^2 is connected, h is onto. It suffices to prove that $g(M^2)$ is (homeomorphic to) M^2 .

Each component K of $\text{bdy}(M^2)$ is a simple closed curve; by the acyclic condition, $f(K)$ is a nondegenerate continuum. Since $\dim(f(B_\rho)) \leq 0$, it follows from the covering property that $f(K) \subset \text{bdy}(N^2)$; thus $\text{int}(M^2) \supset f^{-1}(\text{int}(N^2))$. Since $f(B_\rho) \cap \text{bdy}(N^2) = \emptyset$, $\text{int}(M^2) = f^{-1}(\text{int}(N^2))$. Hence,

for each $y \in g(\text{int}(M^2))$, $g^{-1}(y) \in \text{int}(M^2)$, so that $g|_{\text{int}(M^2)}$ is acyclic mod 2. Since h is light, $\dim(g(M^2)) \leq 2$ [10, pp. 91-92, Theorem VI. 7]. Since any manifold is an orientable gm (generalized manifold) mod 2, it now follows that $g(\text{int}(M^2))$ is a 2-gm [25, p. 22], and thus is a 2-manifold [26, pp. 271-280]. Hence, $g(M^2)$ is a 2-manifold with boundary (and, in fact, $H^i(M^2; Z_2) \approx H^i(g(M^2); Z_2)$ ($i = 0, 1, 2$) by the Vietoris Mapping Theorem [2]).

Since B_h consists of a finite set of points [24, p. 198, (5.1)], there exists a finite set of mutually disjoint closed 2-cells E_j ($j = 1, 2, \dots, m$) such that $\text{bdy}(E_j) \cap f(B_\beta) = \emptyset$, $f(B_\beta) \subset \bigcup_{j=1}^m \text{int}(E_j)$, and each component of $h^{-1}(E_j)$ is again a closed 2-cell containing at most one point of B_h [24, p. 198]. If L is a component of $h^{-1}(E_j)$, then $g^{-1}(L)$ is a 2-manifold with boundary and $g|_{g^{-1}(L)}$ is acyclic mod 2. By [2], $H^1(g^{-1}(L); Z_2) = 0$, so that $g^{-1}(L)$ is a closed 2-cell. Thus M^2 is homeomorphic to $g(M^2)$.

2.3. LEMMA. *Let M^n and N^n be compact connected n -manifolds, possibly with boundary, and let $n \geq 2$. Let $f: M^n \rightarrow N^n$ be a map with $\dim(f(B_\beta)) \leq 0$, $f(B_\beta) \subset \text{int}(N^n)$, and $f^{-1}(y)$ having at most k components for each $y \in N^n$. Then for all but at most $k + r$ points y , where $r = \dim(H^{n-1}(M^n; Z_p))$ and p is any prime, each component A of $f^{-1}(y)$ has $H^{n-1}(A; Z_p) = 0$ (and, if $n = 2$, is acyclic mod p). If $\text{bdy}(N^n) \neq \emptyset$ and f is not constant, then there are at most r such exceptional points.*

Proof. Suppose that there are (at least) m points y such that

$$H^{n-1}(f^{-1}(y); Z_p) \neq 0.$$

If Γ is the union of these sets $f^{-1}(y)$, then $\dim(H^{n-1}(\Gamma; Z_p)) \geq m$. From the exactness of the cohomology sequence, it follows that

$$\dim(H^n(M^n, \Gamma; Z_p)) \geq m - r.$$

Now $f(\Gamma) \subset f(B_\beta)$, and the hypotheses imply that $f|_{f^{-1}(N^n - f(B_\beta))}$ is a covering map [18, p. 128] of degree at most k (Γ may be M^n itself). Suppose that f is not constant; since $\dim(f(B_\beta)) \leq 0$, each component L of $f^{-1}(N^n - f(\Gamma))$ has $f(L) \cap (N^n - f(B_\beta)) \neq \emptyset$. Hence each component of $f^{-1}(N^n - f(\Gamma))$ is mapped by f onto $N^n - f(\Gamma)$, so that $M^n - \Gamma$ has at most k components and $\dim(H^n(M^n, \Gamma; Z_p)) \leq k$. Thus $m \leq k + r$. The same conclusion is immediate if f is constant.

If $\text{bdy}(N^n) \neq \emptyset$ and f is not constant, then (since $f(B_\beta) \subset \text{int}(N^n)$) each component of $M^n - \Gamma$ meets the (nonempty) set $\text{bdy}(M^n)$ so that $H^n(M^n, \Gamma; Z_p) = 0$. Thus $m \leq r$.

2.4. LEMMA. *Let $F: M^2 \times E^{n-2} \rightarrow N^2 \times E^{n-2}$ be a C^2 map such that $F^2(x, t) = t$, where M^2 and N^2 are compact, connected 2-manifolds (possibly with boundary), $n \geq 3$, $\dim(B_F) \leq n - 3$, and*

$$F(B_F) \subset F(R_{n-2}(F)) \cap (\text{int}(N^2) \times E^{n-2}).$$

Suppose further that $F|[(M^2 \times E^{n-2}) - F^{-1}(F(B_F))]$ is a k -to-1 covering map. If $\pi_1(N^2) = 0$, then F is acyclic mod 2.

(The expression F^2 is the second coordinate of F .)

The lemma is first proved under the additional hypothesis that $H^1(F^{-1}(y, t); Z_2) = 0$ for all $(y, t) \in N^2 \times E^{n-2}$ (yielding F monotone). Then, from this case and (2.3), which shows that "most" points (y, t) satisfy this condition, the general result is finally obtained.

Proof. It follows from [5, (1.1)] that if F_t is the map $F|(M^2 \times \{t\})$, then

$$F(R_{n-2}(F)) \cap (N^2 \times \{t\}) = F_t(R_0(F))$$

for all $t \in E^{n-2}$, and thus [3, (1.3)] has dimension at most 0. Hence

(1) $\dim(F(B_F) \cap (N^2 \times \{t\})) \leq 0$ for each $t \in E^{n-2}$.

Note that the branch set of F_t is contained in B_F . From (1) and the covering property of F ($k \geq 1$) it follows that

(2) each F_t is onto.

Suppose that D is a closed 2-cell in N^2 and, for some $s \in E^{n-2}$ and $\nu > 0$, $F(B_F) \cap ((\text{bdy } D) \times S(s, \nu)) = \emptyset$; let K be a component of $F^{-1}(D \times S(s, \nu))$. From the covering property of F it follows that each component of $F^{-1}(\text{bdy } D \times S(s, \nu))$ is homeomorphic to $S^1 \times E^{n-2}$, where S^1 is a circle; and from (2) that $F(K) = D \times S(s, \nu)$. As a result

(3) there is a connected 2-manifold-with-boundary L^2 and a homeomorphism μ of K onto $L^2 \times S(s, \nu)$ such that the restriction

$$\mu|(K \cap (M^2 \times \{t\})) = L^2 \times \{t\}$$

for each $t \in S(s, \nu)$.

First case. Suppose that $H^1(F^{-1}(y, t); Z_2) = 0$ for each $(y, t) \in N^2 \times E^{n-2}$. Thus, each component of $F^{-1}(y, t)$ is acyclic mod 2. In this case it suffices to prove that F is monotone.

Let $H_t G_t$ be the factorization of F_t given by (2.2) (cf. (2)). Suppose that, for some $t \in E^{n-2}$, H_t has no branch point. Since H_t is a covering map [18, p. 128] and $\pi_1(N^2) = 0$, H_t is a homeomorphism; thus F_t is monotone and $k = 1$. From (1) and the covering property of F , it follows that F_s is also monotone for each $s \in E^{n-2}$. Thus,

(4) if, for some $t \in E^{n-2}$, H_t has no branch point, then F is monotone, the desired conclusion.

It follows from [24, p. 198] that H_t is topologically equivalent to a simplicial map. By [22] the number $\lambda(t)$ of branch points of H_t is at most $k\chi(N^2) - \chi(M^2)$, where χ is the Euler characteristic. Choose $\bar{t} \in E^{n-2}$ such that $\lambda(\bar{t})$ is a maximum. By (1) there exists a mutually disjoint finite family of closed 2-cells $D_j \subset \text{int}(N^2)$ ($j = 1, 2, \dots, J$) such that

(a) $F(B_F) \cap (\text{bdy}(D_j) \times \{\bar{t}\}) = \emptyset$,

(b) $F(B_F) \cap (N^2 \times \{t\}) \subset \bigcup_{j=1}^J (\text{int}(D_j) \times \{\bar{t}\})$, and

(c) each component of $H_i^{-1}(D_j \times \{\bar{t}\})$ is a closed 2-cell containing at most one branch point of H_i [22, p. 198]. (Note that F_i , and thus H_i , is locally a homeomorphism on a neighborhood of $F_i^{-1}(\text{bdy}(N^2) \times \{t\})$.) From (c) and (2.2) applied to $F_i|F_i^{-1}(D_j \times \{\bar{t}\})$, each component of $F_i^{-1}(D_j \times \{t\})$ is a closed 2-cell.

There exists a $\delta > 0$ such that each t in $S(\bar{t}, \delta)$ also satisfies condition (a) and (b). Let K_m ($m = 1, 2, \dots, \lambda(\bar{t})$) be those components of $F^{-1}(D_j \times S(\bar{t}, \delta))$ ($j = 1, 2, \dots, J$) such that $G_i(K_m \cap (M^2 \times \{\bar{t}\}))$ has a branch point of H_i (and only one by (3) and condition (c)). By (4) we may suppose that H_i has a branch point, i.e., $\lambda(\bar{t}) \geq 1$. Since $\pi_1(D_j) = 0$, the argument of (4) may be applied to the restriction $F|K_m$. Since $F|K_m$ is not monotone, $G_i(K_m \cap (M^2 \times \{t\}))$ has at least one branch point of H_i for each $t \in S(\bar{t}, \delta)$; by the maximality of $\lambda(\bar{t})$, it has precisely one ($m = 1, 2, \dots$).

Let D_j be the 2-cell such that $F(K_1) = D_j \times S(\bar{t}, \delta)$, and for each $t \in S(\bar{t}, \delta)$ let $\alpha(t)$ be the image under H_i of the (unique) branch point of H_i in $G_i(K_1 \cap (M^2 \times \{t\}))$. The function

$$\alpha: S(\bar{t}, \delta) \rightarrow \text{int}(D_j) \times S(\bar{t}, \delta)$$

is 1-to-1, and (since the union of the branch sets of the H_i is closed) α is continuous. Thus the image of α is a (tame) $(n-2)$ -cell Ω . Let Ψ be $F^{-1}(\Omega) \cap K_1$, let $\hat{\Omega}$ and $\hat{\Psi}$ be the one-point compactifications of these spaces, and let $\zeta: \hat{\Psi} \rightarrow \hat{\Omega}$ be the natural extension of $F|_{\Psi}$. By our assumption, ζ is acyclic mod 2, and, by the Vietoris Mapping Theorem [2], $H_{n-2}(\hat{\Psi}; Z_2) = Z_2$. Thus $\dim \Psi \geq n-2$ [10, p. 137, (F)], so that $\dim(B_F) \geq n-2$, yielding a contradiction. Thus F is monotone, and hence acyclic mod 2.

Second case. Thus, we may suppose that *there exists a point* $(y, t) \in N^2 \times E^{n-2}$ *such that* $H^1(F^{-1}(y, t); Z_2) \neq 0$. By (2.3), for each $t \in E^{n-2}$ there are at most $k+r$ such points $y \in N^2$, where r is the dimension of $H^1(M^2; Z_2)$. Choose \bar{t} such that the number of such points is maximal, and call the points y_i ($i = 1, 2, \dots, m; m \geq 1$). There exist mutually disjoint closed 2-cells D_i such that $y_i \in \text{int}(D_i)$ and (by (1))

$$F(B_F) \cap (\text{bdy}(D_i) \times \{t\}) = \emptyset.$$

Choose $\delta > 0$ such that

$$F(B_F) \cap (\text{bdy}(D_i) \times S(\bar{t}, \delta)) = \emptyset \quad (i = 1, 2, \dots, m).$$

Let K_i be a component of $F^{-1}(D_i \times S(\bar{t}, \delta))$ such that

$$H^1(F^{-1}(y_i, \bar{t}) \cap K_i; Z_2) \neq 0,$$

and let $K_i(t) = K_i \cap F^{-1}(D_i \times \{t\})$. Suppose that for some i and $s \in S(\bar{t}, \delta)$, each $y \in \text{int}(D_i)$ has $H^1((F|K_i)^{-1}(y, s); Z_2) = 0$. Let f be the restriction $F|K_i(s)$; by (2), f is nonconstant. Let hg be the factorization given by (2.2).

Suppose first that h has no branch points. It follows as in (4) that f is

monotone, and (by (2.2)) $K_i(s)$ is a closed 2-cell. From (3), $K_i(\bar{t})$ is also a closed 2-cell, and from (2.3) applied to the restriction map $F|K_i(\bar{t})$, a contradiction of the choice of K_i results.

Thus we may suppose that there exists $x \in B_h$. There exists a closed 2-cell $E \subset D_i$ such that $h(x) \in E \times \{s\}$ and the component E' of $h^{-1}(E \times \{s\})$ containing x is a closed 2-cell containing no other point of B_h [24, p. 198]; by (1), we may suppose that $F(B_F) \cap ((\text{bdy } E) \times \{s\}) = \emptyset$. By (2.2) applied to $F|g^{-1}(E')$, $g^{-1}(E')$ is a closed 2-cell. Choose $\xi > 0$ such that $S(s, \xi) \subset S(\bar{t}, \delta)$ and $F(B_F) \cap ((\text{bdy } E) \times S(s, \xi)) = \emptyset$. Let U be the component of $F^{-1}(E \times S(s, \xi))$ containing $g^{-1}(E')$; then $U \cap (M^2 \times \{t\})$ is a closed 2-cell for each $t \in S(s, \xi)$ (by (3)). It follows from (2.3) that each component in U of $f^{-1}(y, t)$ is acyclic mod 2 for $y \in E$. By the first part of this proof, $F|U$ is monotone, contradicting the choice of E and E' .

(5) Thus, there is no such s ; hence for each $t \in S(\bar{t}, \delta)$, there is at least one point $\alpha_i(t) \in \text{int}(D_i) \times \{t\}$ with $H^1(K_i \cap F^{-1}(\alpha_i(t)); Z_2) \neq 0$.

By the choice of \bar{t} , there is precisely one. The function

$$\alpha_i: S(\bar{t}, \delta) \rightarrow \text{int}(D_i) \times S(\bar{t}, \delta)$$

is 1-to-1 and continuous. Thus, if D_i is assumed to be the closed unit disk, then there is a homeomorphism ϕ of $D_i \times S(\bar{t}, \delta)$ onto itself with $\phi^2(x, t) = t$ and $\phi(\alpha(S(\bar{t}, \delta))) = \{0\} \times S(\bar{t}, \delta)$.

Let P and Q be the one-point compactifications of K_i and $K_i \cap (\phi F)^{-1}(\{0\} \times S(\bar{t}, \delta))$, respectively. It is immediate that

$$H^{n-1}(P; Z_2) \approx H^{n-1}(P, \{p\}; Z_2),$$

where p is the added point; since K_i is homeomorphic to $L^2 \times E^{n-2}$ (by (3)), $H^{n-1}(P, \{p\}; Z_2)$ is isomorphic to $H^{n-1}(L^2 \times S^{n-2}, L^2 \times \{z\}; Z_2)$, where z is any point of S^{n-2} . By the exactness of the cohomology sequence ($n \geq 3$), this is mapped onto $H^{n-1}(L^2 \times S^{n-2}; Z_2)$, which, by the Künneth Formula, is isomorphic to $H^1(L^2; Z_2)$. From (5) and (2.3) it follows that $H^1(L^2; Z_2) \neq 0$; thus $H^{n-1}(P; Z_2) \neq 0$.

Given any closed 2-cell $E \subset D_i - \{0\}$ with $[(\text{bdy } E) \times \{t\}] \cap \phi F(B_F) = \emptyset$, there exists $\xi > 0$ such that, for each $s \in S(t, \xi)$, $[(\text{bdy } E) \times \{s\}] \cap \phi F(B_F) = \emptyset$ also. Since $H^1((\phi F)^{-1}(y, s); Z_2) = 0$ for each $y \in D_i - \{0\}$, it follows from the first part of this proof that

(6) F is monotone on each component of $K_i \cap (\phi F)^{-1}(E \times S(t, \xi))$.

Let V be a closed 2-cell such that $V \subset \text{int}(D_i)$, $0 \in \text{int } V$, and

$$[(\text{bdy } V) \times \{t\}] \cap \phi F(B_F) = \emptyset,$$

and let W be a component of $K_i \cap (\phi F)^{-1}((D_i - \text{int } V) \times \{t\})$. In the factorization given by (2.2) for $f = \phi F|W$, h has no branch points (from (6)), and so is a finite-to-one covering map [16, p. 128]. Since each finite covering space of the annular region $(D_i - \text{int } V) \times \{t\}$ is itself, (2.2)

implies that W is homeomorphic to $(D_i - \text{int } V) \times \{t\}$. Since V may be chosen arbitrarily small, each component of $K_i \cap (\phi F)^{-1}((D_i - \{0\}) \times \{t\})$ is homeomorphic to $D_i - \{0\}$. It follows that $P - Q$ is homeomorphic to $(D_i - \{0\}) \times S(\bar{t}, \delta)$. Since $H^{n-1}(P; Z_2) \neq 0$, $H^{n-1}(Q; Z_2) \neq 0$ from the continuity of Čech cohomology.

Thus, $\dim Q \geq n - 1$ [10, p. 137, (F)], and $K_i \cap (\phi F)^{-1}(\{0\} \times S(\bar{t}, \delta))$ has dimension at least $n-1$. On the other hand, since

$$H^1(K_i \cap (\phi F)^{-1}(0, t); Z_2) \neq 0$$

for all $t \in S(\bar{t}, \delta)$ (by (5) and the definition of ϕ), $K_i \cap (\phi F)^{-1}(\{0\} \times S(\bar{t}, \delta)) \subset B_i$, yielding a contradiction.

2.5. LEMMA. *Let N^p be a C^∞ manifold, and let Γ^{p-q} be a C^r submanifold of N^p ($r = 1, 2, \dots$). Then there is a C^r diffeomorphism ψ of N^p onto itself such that $\psi(\Gamma^{p-q})$ is a C^∞ submanifold of N^{p-q} and ψ is arbitrarily near (in the fine C^r topology) the identity on N^p .*

Proof. The fine C^r topology is defined in [15, p. 25 and p. 28]. (The C^r topology of [16] is the coarse C^r topology of [15].) In [15, p. 35, (4.1)] a slight modification of the proof shows that f_1 may be chosen to approximate f in the C^r topology, i.e., the partial derivatives of f_1 of orders at most r approximate those of f .

In the proof of [15, p. 40, (4.7)] let

$$\psi(x) = x + \bar{g}(\pi(x)) - g(\pi(x))$$

for $x \in \pi^{-1}(\mathcal{O})$, and $\psi(x) = x$ elsewhere. If δ is sufficiently small, then ψ is a C^r diffeomorphism near the identity [15, p. 29, (3.10)]; since $\psi g = \bar{g}$, it follows that the $\psi f = h$.

The remainder of the proof is an analog of that of [15, pp. 41, 42, (4.8) and Exercise (a)], where $f_j = \psi_j f_{j-1}$ ($j = 1, 2, \dots$) and the diffeomorphism ψ is the (well-defined) limit, as $j \rightarrow \infty$, of the composition $\psi_j \psi_{j-1} \dots \psi_2 \psi_1$.

Morse in [14] proved (2.5) in the special case that $N^p = E^p$, Γ^{p-q} is compact, and $q = 1$. Assuming only that N^p is a C^r manifold, Cerf [3, p. 260] proved that there exist a C^∞ manifold Q^p and a C^r diffeomorphism λ of N^p onto Q^p such that $\lambda(\Gamma^{p-q})$ is a C^∞ submanifold of Q^p .

2.6. REMARK. The differentiability hypothesis in [22, p. 26, Theorem I. 5] can be changed from C^n to $C^{\max(n-q+1, 1)}$. (In the proof, replace the theorem of A. P. Morse [22, p. 240] by Sard's Theorem [20].) Moreover, given any m ($m = 1, 2, \dots$) the diffeomorphism A in [22, p. 26] may be chosen C^m .

In [22, p. 27] the hypothesis C^1 suffices, and V^n need not be compact if f is proper.

The following remark follows from the proof of [22, p. 27].

2.7. REMARK. *Let $f: M^n \rightarrow N^n$ be C^m and proper, let K^p be a compact C^m*

p -manifold, and let ρ be a C^m diffeomorphism of a region in N^n onto $K^p \times E^{n-p}$ ($m, p = 1, 2, \dots$). If f is transverse regular [22, pp. 22-23] on $\rho^{-1}(K^p \times \{0\})$, then there are a C^∞ p -manifold $L^p, \epsilon > 0$, and a C^m diffeomorphism σ of $L^p \times S(0, \epsilon)$ onto $f^{-1}(\rho^{-1}(K^p \times S(0, \epsilon)))$ such that $(\rho\sigma)^2(z, r) = r$ for each $r \in S(0, \epsilon)$.

Proof. Since f is transverse regular on $\rho^{-1}(K^p \times \{0\})$, $f^{-1}(\rho^{-1}(K^p \times \{0\}))$ is a C^m p -manifold in M^n [22, p. 23] (the hypothesis C^1 suffices). By [15, p. 42, (4.9)] and [3, p. 260] there is a C^m diffeomorphism ψ of M^n onto a C^∞ n -manifold Q^n such that $\psi(f^{-1}(\rho^{-1}(K^p \times \{0\})))$ is a C^∞ submanifold, call it L^p , of Q^n . Choose $\epsilon > 0$ so that $f\psi^{-1}$ is transverse regular on $\rho^{-1}(K^p \times \{r\})$ for each $r \in S(0, \epsilon)$ [22, p. 27]; thus [20, p. 26] $\psi f^{-1}(\rho^{-1}(K^p \times \{r\}))$ is a C^m p -manifold. By the proof of [22, p. 27], for each point x of L^p , the normal $(n-p)$ -plane to L^p at x meets $\psi(f^{-1}(\rho^{-1}(K^p \times \{r\})))$ in precisely one point, call it $\tau(x, r)$; thus τ is a C^m diffeomorphism of $L^p \times S(0, \epsilon)$ onto $\psi f^{-1}(\rho^{-1}(K \times S(0, \epsilon)))$. Let $\sigma = \psi^{-1}\tau$. (For each $\bar{x} \in f^{-1}(\rho^{-1}(K^p \times S(0, \epsilon)))$, there is a C^m diffeomorphism μ of a neighborhood V of $f(\bar{x})$ onto E^n such that $\mu(V \cap \rho^{-1}(K^p \times \{r\}))$, for each $r \in E^{n-p}$, is defined by x^i constant ($i = 1, 2, \dots, n-p$). Perhaps it is clearer that both σ and σ^{-1} are C^m if we observe [5, (1.1)] and [6, (2.3)] that there is a neighborhood U of \bar{x} and a C^m diffeomorphism λ of U onto E^n such that the map $g = \mu\lambda^{-1}$ satisfies $g^i(x^1, x^2, \dots, x^n) = x^i$ ($i = 1, 2, \dots, n-p$).)

2.8. LEMMA. Let M^n and N^n be connected C^3 n -manifolds, and let $f: M^n \rightarrow N^n$ be C^3 and proper, with $n \geq 4$, $\dim(B_f) \leq n-3$ and $f(B_f) \subset f(R_{n-2}(f))$. Then f satisfies hypotheses (1) and (2) of (2.1).

Proof. By [19, §5], $\dim(f(R_{n-2})) \leq n-2$, so that $\dim(f^{-1}(f(R_{n-2}))) \leq n-2$. Since f is proper, and R_{n-2} is locally compact, $f(R_{n-2})$ is locally compact. The restriction map $f| [M^n - f^{-1}(f(R_{n-2}))]$ is a k -to-1 covering map [18, p. 128] with connected domain, so that f satisfies (2). Suppose that, for some point y in N^n , $f^{-1}(y)$ has at least $k+1$ components Y_i ($i = 1, 2, \dots, k+1$). There is a connected open neighborhood W about y such that the components Y_i are contained in different components of $f^{-1}(W)$. A contradiction results from (2) and the covering property. Thus, for each $y \in N^n$, $f^{-1}(y)$ has at most k components.

If, for each y in N^n , there is a Euclidean coordinate neighborhood E about y such that $f^{-1}(E)$ has exactly k components, then, by (2), condition (1) is satisfied. Thus we may suppose that $N^n = E$ and $k \neq 1$, and deduce a contradiction. By [15, p. 41] we may suppose that M^n and N^n are C^∞ manifolds, and thus that $E^n = E$ ($= N^n$).

The homomorphism f_* of $\pi_1(M^n - f^{-1}(f(R_{n-2})))$ into $\pi_1(E^n - f(R_{n-2}))$ induced by $f| [M^n - f^{-1}(f(R_{n-2}))]$ is one-to-one but not onto [9, pp. 93 and 96]. Since the semi-linear maps of the circle S^1 into $E^n - f(R_{n-2})$

generate $\pi_1(E^n - f(R_{n-2}))$ and $n \geq 3$, the (polyhedral) embeddings of S^1 also generate $\pi_1(E^n - f(R_{n-2}))$. Let γ be an embedding such that homotopy class of γ is not in $f_*(\pi_1(M^n - f^{-1}(f(R_{n-2}))))$; each component of $f^{-1}(\gamma(S^1))$ is a topological circle (by the covering property), and each is mapped by f onto $\gamma(S^1)$ nonhomeomorphically. Moreover, there exists $\epsilon > 0$ such that, if λ is any embedding of S^1 in E^n with (uniform) distance $d(\lambda, \gamma) < \epsilon$, then $\lambda(S^1)$ has the same property.

By [8, p. 111, Theorem 2a], there is a C^∞ embedding λ of S^1 in E^n ($n \geq 4$) such that $d(\lambda, \gamma) < \epsilon/2$. By [8, p. 110, Corollary] there is a C^∞ embedding μ of the unit 2-disk D^2 in E^n which extends λ . If D^j denotes the unit j -disk in $E^j \subset E^n$, $D^2 \subset D^3 \subset D^n$, there is [15, p. 275] a C^∞ embedding ν of D^n into E^n which extends μ . Since $\text{bdy}(D^2) \subset \text{bdy}(D^3) = S^2$, $\xi = \nu|_{S^2}$ is a C^∞ embedding which extends λ .

Let A be the (C^3) map given by [22, p. 26] (cf. 2.6), with (uniform) distance from the identity map less than $\epsilon/2$, for f and $N^{p-q} = \xi(S^2)$. By the proof of the theorem, f is transverse regular [22, p. 22-23] on $A^{-1}(\xi(S^2))$, call it X^2 . Because ξ is the restriction of ν , X^2 has a tubular neighborhood Z and a C^3 diffeomorphism ρ of Z onto $X^2 \times E^{n-2}$.

By 2.7, if Z is a sufficiently small tubular neighborhood, the restriction of f to each component U of $f^{-1}(Z)$ satisfies the hypotheses of 2.4; thus the restriction $f|_U$ is monotone. Because of the choice of $A^{-1}(\lambda(S^1))$, each component of $f^{-1}(A^{-1}(\lambda(S^1)))$ is a topological circle mapped by f onto the topological circle $A^{-1}(\lambda(S^1))$ nonhomeomorphically; from condition (2), $f(U) = Z$, and a contradiction results. Thus f does satisfy hypothesis (1) of 2.1.

3. The proof of (1.1). Given $U \subset E^n$ and a C^1 map $h: U \rightarrow E^1$, $D_j h$ denotes the first partial derivative of h with respect to the j th independent variable. If $n = 1$ and h is C^j , then $D^{(j)} h$ is the derivative of order j ($D^{(0)} h = h$).

3.1. LEMMA. *Let $f: M^n \rightarrow N^n$ be C^m and proper, with $\dim(B_f) \leq n - 2$ and $m = 2, 3, \dots$. Suppose that the Jacobian matrix (derivative map) has rank at least $n-1$ at every point of M^n . Then,*

(1) *for each point y in N^n , $f^{-1}(y)$ has a finite number of components, each of which is either a point or a C^m embedding of a closed interval. Moreover,*

(2) *for each open neighborhood U of y , there is a region W such that $y \in W \subset U$ and each component of $f^{-1}(W)$ which meets $f^{-1}(y)$ is mapped by f onto W .*

Proof. We may suppose that $n \geq 2$ [18, p. 128], M^n and N^n are C^∞ Riemannian manifolds [15, p. 42, (4.9)], and, in fact, that $N^n = E^n$. For each point $\bar{x} \in f^{-1}(y)$, there are [5, (1.1)] C^m diffeomorphisms λ of a Euclidean neighborhood $U(\bar{x})$ of \bar{x} onto E^n and μ of E^n onto itself (the latter merely interchanging dependent variables [5, (2.3)]) such that the map $g = \mu f \lambda^{-1}$ has $g^i(x^1, x^2, \dots, x^n) = x^i$ ($i = 1, 2, \dots, n-1$). Since $\dim(B_f) \leq n - 2$,

the Jacobian determinant of g is either non-negative or nonpositive. Thus $D_n g^n(x^1, x^2, \dots, x^n) \geq 0$ (say) on all of $\lambda(U(\bar{x}))$, so that the map $h: E^1 \rightarrow E^1$ defined by $g^n(x^1, x^2, \dots, x^n) = h(x^n)$ is monotone, for all $(x^1, x^2, \dots, x^n) \in E^n$. As a result, since f is proper, $f^{-1}(y)$ has a finite number of components, each a point, a C^m embedding of a closed interval, or a C^m embedding of the circle S^1 .

Suppose that Ω is any component of $f^{-1}(y)$ and that U is any open set containing y . There is an open n -cell Y , $y \in Y$ and $\bar{Y} \subset U$, such that each component of $f^{-1}(\bar{Y})$ meets at most one component of $f^{-1}(y)$; let X be the component of $f^{-1}(Y)$ which contains Ω . There is a C^m embedding α of a 1-manifold M^1 in X such that either $M^1 = S^1$ and $\Omega = \alpha(M^1)$ or $M^1 = E^1$ and $\Omega \subset \alpha(M^1)$. By 2.5 we may suppose that $\alpha(M^1)$ is a C^∞ submanifold of X .

Let $N(\delta)$ consist of the vectors with length at most δ in the normal bundle to $\alpha(M^1)$. For δ sufficiently small the restriction of the exponential map E to $N(\delta)$ is a C^∞ diffeomorphism onto a (tubular) neighborhood of $\alpha(M^1)$. For each $x \in \Omega$, let $D(x)$ be the image under E of the $(n-1)$ -disk of vectors normal to $\alpha(M^1)$ at x . From the coordinate representation at x given above, observe that any set of $(n-1)$ -vectors spanning an $(n-1)$ -plane transversal to $\alpha(M^1)$ at x is mapped by the Jacobian matrix of f onto a set of $n-1$ independent vectors. Thus the Jacobian matrix of the restriction map $f|D(x)$ has maximal rank at x . Moreover, given any $\bar{x} \in \Omega$, if an interval from x to \bar{x} in Ω is contained in the coordinate neighborhood $U(\bar{x})$, then it follows from the form of f on $U(\bar{x})$ (the fact that $D_n g^n \geq 0$) that $f(D(\bar{x}))$ and $f(D(x))$ have the same tangent $(n-1)$ -plane at y . Thus, for all $x \in \Omega$, the sets $f(D(x))$ have the same tangent $(n-1)$ -plane at y , which we may suppose is that spanned by the first $n-1$ coordinate vectors of E^n .

There is a C^∞ embedding β of S^1 in Y such that (a) $y \in \beta(S^1)$ and (b) $\beta(S^1)$ is normal to that plane at y . We have observed above that, for any such embedding β , f is transverse regular [22, pp. 22-23] to $\beta(S^1)$ at each point of Ω . In [22, pp. 24-26] (cf. 2.6), the space H of C^m maps A may be replaced by that subspace such that $A(y) = y$ and the first partial derivatives of A agree with those of the identity map at y . As a result, there is a C^m embedding γ of S^1 in Y such that it satisfies conditions (a) and (b), and f is transverse regular at each point of $f^{-1}(\gamma(S^1) - \{y\})$. Thus $f|X$ is transverse regular at each point of $f^{-1}(\gamma(S^1))$. By 2.5 and [22, p. 27], we may suppose that $\gamma(S^1)$ is a C^∞ submanifold of Y .

Define, as above, a tubular neighborhood T of $\gamma(S^1)$, $T \subset Y$. Then there is a C^∞ diffeomorphism ρ of T onto $\gamma(S^1) \times E^{n-1}$ such that $\rho(z) = (z, 0)$ for each $z \in \gamma(S^1)$ (for $n = 2$, the Moebius Band cannot be embedded in E^2 ; for $n \geq 3$, by the proof of [11, (1.2)]). Let L^1 , $\epsilon > 0$, and σ be as given

by 2.7 for $f|(X \cap f^{-1}(T))$ and $K^p = \gamma(S^1)$, and let R be the component of $X \cap f^{-1}\rho^{-1}(\gamma(S^1) \times S(0, \epsilon))$ containing Ω ; we will still denote $R \cap f^{-1}\rho^{-1}(\gamma(S^1) \times \{0\})$ by L^1 , a 1-sphere by 2.7.

Suppose that Ω is a 1-sphere, i.e., that $f(L^1) = \{y\}$. Given any closed interval Γ of $\gamma(S^1)$, $y \in \text{int}\Gamma$, there exists a sufficiently small neighborhood $S(0, \xi) \subset S(0, \epsilon)$ such that $\rho f \sigma(L^1 \times S(0, \xi)) \subset \Gamma \times S(0, \epsilon)$. Because of the form of f on the neighborhoods $U(x)$ ($D_n g^n \geq 0$), it follows that $\rho f \sigma(L^1 \times \{r\})$ is a single point, for each $r \in S(0, \xi)$, contradicting the fact that $\dim(B_\beta) < n$. Thus Ω is not a 1-sphere, yielding conclusion (1).

From the local form of f ($D_n g^n \geq 0$) and the fact that no point inverse is a 1-sphere, $\rho f \sigma|(L^1 \times \{r\})$ is the composition of a monotone map and a finite-to-one covering map, and $\rho f \sigma(L^1 \times \{r\}) = \gamma(S^1) \times \{r\}$. Since $\Omega (= f^{-1}(y) \cap R)$ is connected, each map $\rho f \sigma|(L^1 \times \{r\})$ is monotone; thus $f|R$ is monotone, and $f(R) = T$.

For each Ω_i in the finite set of components of $f^{-1}(y)$, there is an open tubular neighborhood T_i of y satisfying the conclusion of 2.7, $T_i \subset Y \subset U$; let R_i be the component of $f^{-1}(T_i)$ containing Ω_i . Since $T_i \subset Y$, the sets R_i are mutually disjoint. By the above paragraph, $f|R_i$ is monotone and $f(R_i) = T_i$. Let W be the component of $\bigcap_i T_i$ containing y . Then $f(R_i \cap f^{-1}(W)) = W$, and $f|(R_i \cap f^{-1}(W))$ is monotone. By [24, p. 138, (2.2)] (it is sufficient that f be proper), $R_i \cap f^{-1}(W)$ is connected, and conclusion (2) follows.

3.2. COROLLARY. *Let $f: M^n \rightarrow N^n$ be a C^2 proper map with $\dim(B_\beta) \leq n - 2$. Then f satisfies condition (2) of (2.1).*

Proof. Given a region $U \subset N^n - f(R_{n-2})$, let V be a component of $f^{-1}(U)$. Suppose that $f(V) \neq U$. Since $V \subset \bar{V} \cap f^{-1}(U) \subset \bar{V}$, $\bar{V} \cap f^{-1}(U)$ is connected and thus is contained in V . Since f is proper it follows that $f(V) = \text{Cl}[f(V)] \cap U$. Thus there exists $y \in U \cap \text{bdy}(f(V))$. Let W be the region given by 3.1, conclusion (2), for U and y . Since $y \in f(V)$, $W \subset f(V)$, and a contradiction results; thus $f(V) = U$.

Since $\dim(f(R_{n-2})) \leq n - 2$ [19, §5] and $\dim(B_\beta) \leq n - 2$, $\dim(f^{-1}(f(R_{n-2}))) \leq n - 2$; that f satisfies conclusion (2) of 2.1 for arbitrary regions $U \subset N^n$ follows.

3.3. LEMMA. *Let E_+^n , E_-^n , and Δ be the sets in E^n defined by $x^n > 0$, $x^n < 0$, and $-\lambda < x^i < \lambda$ ($i = 1, 2, \dots, n-1$) and $-\mu < x^n < \nu$ (λ, μ , and $\nu > 0$), respectively. Let U be an open set with $\bar{\Delta} \subset U \subset E^n$, let $\epsilon > 0$, and let $f: U \rightarrow E^n$ be C^m ($m = 1, 2, \dots$) with $f^i = x^i$ ($i = 1, 2, \dots, n-1$) on $\bar{\Delta}$, $D_n f^n \geq 0$ on $\bar{\Delta}$, and $D_n f^n > 0$ on $\bar{\Delta} \cap \bar{E}_+^n$. Then there is a C^m map $g: U \rightarrow E^n$ such that*

- (i) $f = g$ off Δ ,
- (ii) the (uniform) distance between each partial derivative of f with order $0, 1, \dots, m$ and the corresponding partial of g is at most ϵ ,
- (iii) the Jacobian determinant $J(g) > 0$ on $\bar{\Delta}$,

(iv) $B_g \subset B_f$, and

(v) $g(\bar{\Delta}) = f(\bar{\Delta})$. Moreover,

(vi) for each y in $f(\bar{\Delta})$, $\bar{\Delta} \cap f^{-1}(y)$ and $\bar{\Delta} \cap g^{-1}(y)$ each have one component.

Proof. We may suppose that $\bar{\Delta}$ is defined by $0 < x^i < 1$ ($i = 1, 2, \dots, n-1$) and $-1 < x^n < 1$, that $\epsilon < 1$, and that ϵ is less than the minimum of $D_n f^n$ on $\bar{\Delta} \cap \bar{E}_+$. There exists a C^m map $\alpha: E^1 \rightarrow E^1$ such that $\alpha \geq 0$, $\alpha > 0$ precisely on $(0, 1)$, and $D^{(j)}\alpha < \epsilon$ ($j = 0, 1, \dots, m$). Let $\beta_k: E^n \rightarrow E^1$ be defined by

$$\beta_k(x^1, x^2, \dots, x^n) = \alpha((-1)^k x^n) \prod_{i=1}^{n-1} \alpha(x^i) \quad (k = 1, 2),$$

let $\gamma = \beta_1$ for $x^n \leq 0$ and $\gamma = -\beta_2$ for $x^n \geq 0$, and let

$$\delta = \int_{-1}^{x^n} \gamma(x^1, x^2, \dots, x^{n-1}, t) dt.$$

Let $g^i = f^i$ ($i = 1, 2, \dots, n-1$), and $g^n = f^n + \delta$.

Since $\gamma > 0$ on $\Delta \cap E_-^n$ and $D_n \delta = \gamma$, $D_n g^n > 0$ on $\Delta \cap E_-^n$; since $|\gamma| < \epsilon$ on $\Delta \cap E_+^n$ and $\epsilon < D_n f^n$ on $\Delta \cap \bar{E}_+^n$, $D_n g^n > 0$ on $\Delta \cap \bar{E}_+^n$. Conclusion (iii) follows. Since

$$\begin{aligned} \delta(x^1, x^2, \dots, x^{n-1}, -1) &= \delta(x^1, x^2, \dots, x^{n-1}, 1) = 0, \\ f^n(x^1, x^2, \dots, x^{n-1}, -1) &= g^n(x^1, x^2, \dots, x^{n-1}, -1) \end{aligned}$$

and

$$f^n(x^1, x^2, \dots, x^{n-1}, 1) = g^n(x^1, x^2, \dots, x^{n-1}, 1);$$

since $f^i = g^i = x^i$ ($i = 1, 2, \dots, n-1$), and since $D_n f^n \geq 0$ and $D_n g^n \geq 0$ on $\bar{\Delta}$, $g(\bar{\Delta}) = f(\bar{\Delta})$. The reader may verify that g satisfies the remaining conditions.

3.4 LEMMA. Let M^n and N^p be C^∞ manifolds, and let \mathfrak{E}^m be the set of all C^m maps $g: M^n \rightarrow N^p$ ($m = 1, 2, \dots$). Then there are an embedding of N^p as a closed submanifold of a Euclidean space, and a complete metric ρ on \mathfrak{E}^m such that:

- (1) $\rho(g, g_j) \rightarrow 0$ implies that $g_j \rightarrow g$ in the (coarse) C^m topology; and
- (2) if d is the metric induced on N^p by the embedding and $\rho(g, h) < 1$, then $d(g(x), h(x)) \leq \rho(g, h)$ for all $x \in M^n$.

Proof. The coarse C^m topology is defined in [16] (cf [15, pp. 25-28]). Let $T_1(M^n)$ be the tangent bundle of M^n , let $T_j(M^n) = T_1(T_{j-1}(M^n))$, and let $d_m g: T_m(M^n) \rightarrow T_m(N^p)$ be the m th derivative map. Since $T_m(M^n)$ and $T_m(N^p)$ are C^∞ manifolds, each may be embedded as a closed C^∞ submanifold of some Euclidean space [15, p. 20]. For each $x \in M^n$, replace the fiber F_x over x by the unit ball B_x (using the Euclidean metric), B_x

$\subset F_x \subset T_m(M^n)$, defining the space $B(M^n)$. For maps G and H of $B(M^n)$ into $T_m(N^p)$, let $\sigma(G, H)$ be the minimum of 1 and the least upper bound of $\{d(G(x), H(x)) : x \in B(M^n)\}$, where d is the Euclidean metric; then σ is a complete metric. Let $\rho(g, h) = \sigma(d_m g, d_m h)$. From the natural embedding of N^p in $T_m(N^p)$, conclusion (2) follows.

The coarse C^m topology on \mathfrak{E}^m is, similarly, the compact open topology of the m th derivative maps of $B(M^n)$ (in fact, $T_m(M^n)$) into $T_m(N^p)$. Since the compact open topology is the topology of uniform convergence on compact sets [1, p. 485], $\rho(g, g_j) \rightarrow 0$ implies that $g_j \rightarrow g$ in the coarse C^m topology. (In fact, if M^n is compact, the topology of ρ is the (coarse = fine) C^m topology.)

Lastly we wish to show that ρ is complete. Suppose that g_j is a ρ -Cauchy sequence of \mathfrak{E}^m ; then $d_m g_j$ is a σ -Cauchy sequence, and thus has a limit D . Since the restrictions $d_m g_j|_{M^n} \rightarrow D|_{M^n}$ uniformly, i.e., $g_j \rightarrow D|_{M^n}$ uniformly, it suffices to prove that $d_m(D|_{M^n}) = D$. In case $m = 1$ and $M^n = N^p = E^1$, the result is a straightforward consequence of the Mean-Value Theorem, and the general result follows from this special case.

The topology defined above is intermediate between the fine and coarse C^r topologies. In [3, p. 272], Cerf observes that the fine C^r topology (his \mathcal{C}^r [3, p. 269]) is not metrizable, but has a complete uniform structure. The coarse C^r topology is metrizable with a complete metric (use [1, p. 490, Theorem 10] and the metrization theorem for uniform spaces), but, in general, it does not have a metric satisfying condition (2).

3.5. LEMMA. *Let M^n and N^n be connected C^3 manifolds, and let $f: M^n \rightarrow N^n$ be a C^3 proper map with $\dim(B_f) \leq n - 3$ and $n \geq 4$. Then f satisfies condition(1) of (2.1).*

Proof. By [15, p. 42] we may suppose that M^n and N^n are C^∞ manifolds and that \mathfrak{E}^3 , d , and ρ are as in 3.4. By [19, §5], $\dim(f(R_{n-2}(f))) \leq n - 2$, and, since f is proper, $f(R_{n-2}(f))$ is locally compact. There exist compact sets Y_j and open coordinate neighborhoods U_j and V_j such that $Y_j \subset V_j$, $\bar{V}_j \subset U_j$, \bar{U}_j is compact, the \bar{U}_j are locally finite in $N^p - f(R_{n-2}(f))$, and $N^p - f(R_{n-2}(f)) = \bigcup_{j=1}^\infty Y_j = \bigcup_{j=1}^\infty \bar{U}_j$. Let F_j be the set of $x \in M^n$ such that $d(x, B_f) < 1/j$.

Given $h \in \mathfrak{E}^3$ and $x \in M^n$, let $r(h, x)$ be the rank of (the Jacobian matrix of) h at x ; let $0 < \epsilon < 1$. A sequence of proper maps $f_i \in \mathfrak{E}^3$ ($i = 1, 2, \dots$; $f_0 = f$) satisfies property \mathfrak{P} if

- (a) f_i agrees with f_{i-1} off $F_i \cap f^{-1}(V_i)$;
- (b) $f_i(f^{-1}(\bar{V}_j)) \subset U_j$ ($j = 1, 2, \dots$);
- (c) $\bigcup_{j=1}^i f^{-1}(Y_j) \cap B(f_i) = \emptyset$, where $B(f_i)$ is the branch set of f_i ;
- (d) for each $x \in M^n$, $r(f_i, x) \geq r(f_{i-1}, x)$, and either $f_i(x) = f(x)$ or $r(f_i, x) = n$;

- (e) $B(f_i) \subset B(f_{i-1})$;
 (f) $\rho(f_i, f_{i-1}) < 2^{-i}\epsilon$; and
 (g) for each $y \in N^p$, $f_i^{-1}(y)$ and $f_{i-1}^{-1}(y)$ have the same number of components.

We wish to construct a sequence $\{f_i\}$ satisfying \mathfrak{B} . Suppose that f_1, f_2, \dots, f_{i-1} have been defined; by (e) $\dim(B(f_{i-1})) \leq n-3$. There exists η , $0 < \eta < 2^{-i}\epsilon$, such that, if f_i is any C^3 map satisfying (a) and $\rho(f_i, f_{i-1}) < \eta$, then f_i satisfies (b) (and (f)).

Since $r(f_{i-1}, x) \geq n-1$ for $x \in f^{-1}(\bar{V}_i)$ (by (d)) there exists ξ , $0 < \xi < \eta$, such that $h \in \mathfrak{E}^3$, $x \in f^{-1}(\bar{V}_i)$, and $\rho(h, f_{i-1}) < \xi$ implies that $r(h, x) \geq n-1$. For each $\bar{x} \in F_i \cap f^{-1}(V_i)$, let $U[h, \bar{x}]$ and $V[h, \bar{x}]$ be the neighborhoods given by [5, (1.1)], $U[h, \bar{x}] \subset F_i \cap f^{-1}(V_i)$, with diffeomorphisms $k_1[h, \bar{x}]$ of E^n onto $U[h, \bar{x}]$ and $k_2[h, \bar{x}]$ of $V[h, \bar{x}]$ onto E^n such that $k_2 h k_1$, call it H , has the form $H^j = x^j$ ($j = 1, 2, \dots, n-1$). We may suppose that the image of $k_1[h, \bar{x}]$ evaluated at 0 is \bar{x} . Let $\Delta_{\lambda, \mu, \nu} \subset E^n$ be the set of 3.3, let $\bar{\Delta}[h, \bar{x}, \lambda, \mu, \nu]$ be the $k_1[h, \bar{x}]$ image of $\Delta_{\lambda, \mu, \nu}$, and let the top of $\bar{\Delta}[h, \bar{x}, \lambda, \mu, \nu]$ (respectively, bottom; center line; $\bar{\Delta}^+[h, \bar{x}, \lambda, \mu, \nu]$) be the $k_1[h, \bar{x}]$ image of the subset of $\bar{\Delta}_{\lambda, \mu, \nu}$ defined by $x^n = \nu$ (respectively, $x^n = -\mu$; $x^i = 0$, $i = 1, 2, \dots, n-1$; $x^n \geq 0$).

The compact set $\bar{F}_{i+1} \cap f^{-1}(Y_i)$ is contained in the open set $F_i \cap f^{-1}(V_i)$. Given $y \in N^n$ and a component Γ of $f_{i-1}^{-1}(y)$ such that $\Gamma \cap \bar{F}_{i+1} \cap f^{-1}(Y_i) \neq \emptyset$, we will now prove that $\Gamma \subset F_{i+1} \cap f^{-1}(Y_i)$. We may suppose that Γ is nondegenerate; thus $\Gamma \subset B(f_{i-1}) \subset B_j$ (by (e) for f_{i-1}), so that $\Gamma \subset \bar{F}_{i+1}$. By (d) for f_{i-1} , $f|_{\Gamma} = f_{i-1}|_{\Gamma}$, so that $f(\Gamma) = \{y\}$. Since $\Gamma \cap f^{-1}(Y_i) \neq \emptyset$, $\Gamma \subset f^{-1}(Y_i)$, yielding the desired conclusion.

Since $\dim(B(f_{i-1})) \leq n-3$ (by (e)), $r(f_{i-1}, x) \geq n-1$ for each $x \in f^{-1}(V_i)$ (by (d)), and f_{i-1} is proper (by (a)), it follows from 3.1 that for each $y \in N^n$, each component Γ of $f_{i-1}^{-1}(y) \cap \bar{F}_{i+1} \cap f^{-1}(Y_i)$ is a point or a C^3 embedding of an interval. For each $\bar{x} \in \bar{F}_{i+1} \cap f^{-1}(Y_i)$, there exist $\lambda(\bar{x}) > 0$, $\mu(\bar{x}) > 0$, and $\nu(\bar{x}) > 0$ such that

$$\bar{\Delta}[f_{i-1}, \bar{x}, \lambda(\bar{x}), \mu(\bar{x}), \nu(\bar{x})] \subset F_i \cap f^{-1}(V_i).$$

For each $h \in \mathfrak{E}^3$, let $\bar{\Delta}[h, \bar{x}, \lambda(\bar{x}), \mu(\bar{x}), \nu(\bar{x})]$ be denoted by $\bar{\Delta}(h, \bar{x})$.

Let Γ be the component of $f_{i-1}^{-1}(f_{i-1}(\bar{x}))$ containing \bar{x} ; then $\Gamma \cap \bar{\Delta}(f_{i-1}, \bar{x})$ is contained in the center line of $\bar{\Delta}(f_{i-1}, \bar{x})$. If Γ is an arc and $\bar{x} \in \text{int } \Gamma$, we may as well suppose, by replacing $\mu(\bar{x})$ and $\nu(\bar{x})$ by small positive numbers, that $\Gamma \cap \bar{\Delta}(f_{i-1}, \bar{x})$ is that center line. Similarly, if $\Gamma = \{\bar{x}\}$ or if Γ is an arc and \bar{x} is one of the two endpoints of Γ , then Γ is disjoint from either the top or bottom of $\bar{\Delta}(f_{i-1}, \bar{x})$, say the top, and we may suppose that f_{i-1} has rank n on (a neighborhood of) the top of $\bar{\Delta}(f_{i-1}, \bar{x})$.

For each such arc component Γ , there exists a finite number of points x_m ($m = 1, 2, \dots, L(\Gamma)$) such that the sets $\bar{\Delta}(f_{i-1}, x_m)$ are a minimal cover

of Γ . By renumbering the points x_m , replacing $\lambda(x_m)$, $\mu(x_m)$, and $\nu(x_m)$ by smaller numbers, and by (possibly) interchanging tops and bottoms (by a reflection of E^n), we may suppose that

- (i) f_{i-1} has rank n on top of $\Delta(f_{i-1}, x_1)$, and that
- (ii) the top of $\Delta(f_{i-1}, x_{m+1})$ is contained in $\Delta(f_{i-1}, x_m)$ ($m = 1, 2, \dots, L(\Gamma) - 1$).

For Γ an arc component, let $\Omega(\Gamma)$ be the union of these sets $\Delta(f_{i-1}, x_m)$; for Γ the single point \bar{x} , let $\Omega(\Gamma) = \Delta(f_{i-1}, \bar{x})$. The sets $\Omega(\Gamma)$ constitute an open cover of $\bar{F}_{i+1} \cap f^{-1}(Y_i)$, and thus there is a finite subcover. The corresponding sets $\Delta(f_{i-1}, x_j)$ ($j = 1, 2, \dots, J$) thus also constitute a finite subcover, and, for each Γ of the finite subcover, may be ordered consistent with the ordering on Γ . If $\mu' + \nu' = \mu + \nu$ (μ, μ', ν , and $\nu' > 0$), then $\Delta_{\lambda, \mu, \nu}$ is homeomorphic to $\Delta_{\lambda, \mu', \nu'}$ by a translation in the n th coordinate direction. Thus we may suppose that either

- (i) f_{i-1} has rank n on (a neighborhood of) $\bar{\Delta}^+(f_{i-1}, x_j)$ ($j = 1, 2, \dots, J$) or
- (ii) $\bar{\Delta}^+(f_{i-1}, x_j) \subset \Delta(f_{i-1}, x_{j-1})$ ($j = 2, 3, \dots, J$).

The reader may verify from the proofs of [4, (1.1)] and the rank theorem [12, p. 7, (1.8)] that, if $\rho(h_j, f_{i-1}) \rightarrow 0$ as $j \rightarrow \infty$ and $\bar{x} \in \bar{F}_{i+1} \cap f^{-1}(Y_i)$, then we may suppose that $k_1[h_j, \bar{x}]|_{\bar{\Delta}_{\lambda, \mu, \nu}}$ approaches $k_1[f_{i-1}, \bar{x}]|_{\bar{\Delta}_{\lambda, \mu, \nu}}$ in the C^1 topology.

Thus there exists ζ , $0 < \zeta < \xi$, such that $h_j \in C^3$ and $\rho(h_j, f_{i-1}) < \zeta$ ($j = 1, 2, \dots, J$) implies that

(1) the sets $\Delta(h_j, x_j)$ cover $\bar{F}_{i+1} \cap f^{-1}(Y_i)$ and each set $\bar{\Delta}(h_j, x_j) \subset F_i \cap f^{-1}(V_i)$, and

(2) for each j , either

- (i) h_j has rank n on $\bar{\Delta}^+(h_j, x_j)$, or
- (ii) $\bar{\Delta}^+(h_j, x_j) \subset \Delta(h_{j-1}, x_{j-1})$.

Let $h_1 = f_{j-1}$. Let h_2 be the map given by 3.3 such that $\rho(h_1, h_2) < \zeta/2$, h_2 agrees with h_1 off $\Delta(h_1, x_1)$, and h_2 has rank n on $\Delta(h_1, x_1)$. Suppose that h_{j+1} ($j = 1, 2, \dots, m-1$) have been given by 3.3 for h_j with $\rho(h_j, h_{j+1}) < 2^{-j}\zeta$, h_{j+1} agrees with h_j off $\Delta(h_j, x_j)$, and h_{j+1} has rank n on $\Delta(h_j, x_j)$. By (2), either (i) h_m has rank n on $\bar{\Delta}^+(h_m, x_m)$ or (ii) $\bar{\Delta}^+(h_m, x_m) \subset \Delta(h_{m-1}, x_{m-1})$; but, by the inductive hypothesis, h_m has rank n on $\Delta(h_{m-1}, x_{m-1})$. Thus, in either case, h_m has rank n on $\bar{\Delta}^+(h_m, x_m)$. Let h_{m+1} be the map given by 3.3 such that $\rho(h_{m+1}, h_m) < 2^{-m}\zeta$, h_{m+1} has rank n on $\Delta(h_m, x_m)$, and h_{m+1} agrees with h_m off that set.

Let f_i be the map h_{j+1} thus defined. Condition $\mathfrak{P}(a)$ follows from (1) and 3.3(i); $\mathfrak{P}(c)$ from (1) and 3.3(ii); $\mathfrak{P}(b)$ and (f) from the fact that $\zeta < \eta$; $\mathfrak{P}(d)$ from 3.3(i) and (ii); $\mathfrak{P}(e)$ from 3.3(iv); and $\mathfrak{P}(g)$ from 3.3(i) and (vi). Thus there exists a sequence $\{f_i\}$ satisfying \mathfrak{P} .

Since ρ is complete, a limit map $h \in C^3$ exists, and $\rho(f, h) < \epsilon$. Let $X \subset Y$ be any compact set, and let Y consist of those points $y \in N^p$ such

that $d(y, X) \leq \epsilon$. Then Y is closed and bounded, and, since N^n is a closed subset of a Euclidean space (3.4), Y is compact. If $x \in M^n - f^{-1}(Y)$, then $d(f(x), X) > \epsilon$, so that (3.4(2)) $h(x) \notin X$. Thus $h^{-1}(X) \subset f^{-1}(Y)$, and hence is compact. Since X is an arbitrary compact set in N^n , h is proper.

Given $x \notin B_f$, choose j such that $x \notin \bar{F}_j$. By $\mathfrak{P}(a)$, $f_{j-1} = h$ on a neighborhood of x ; by $\mathfrak{P}(e)$, $x \notin B_h$. Thus $B_h \subset B_f$. Given $z \notin f^{-1}(f(R_{n-2}(f)))$, there exists i such that $z \in f^{-1}(Y_i)$; by $\mathfrak{P}(c)$, $z \notin B(f_i)$. There exists J such that $\bar{U}_j \cap \bar{U}_i = \emptyset$ for all $j \geq J$. By $\mathfrak{P}(a)$ and (b) h agrees with f_j on a neighborhood of z . By $\mathfrak{P}(e)$, $z \notin B_h$. Thus $B_h \subset B_f \cap f^{-1}(f(R_{n-2}(f)))$.

From $\mathfrak{P}(e)$ and 3.4(1), for all $x \in M^n$ $d(f_i(x), f_{i-1}(x)) < 2^{-i}\epsilon$ ($0 < \epsilon < 1$); thus $d(f_i(x), h(x)) < 2^{-i}\epsilon$. From $\mathfrak{P}(a)$ and $\mathfrak{P}(b)$, there exists a neighborhood W of $f^{-1}(f(R_{n-2}(f)))$ such that, for all $x \in W$, $d(f(x), h(x)) < 2^{-i}\epsilon$. Since i is arbitrary, the restriction map $h|f^{-1}(f(R_{n-2}(f))) = f|f^{-1}(f(R_{n-2}(f)))$. Similarly, all partial derivatives with order at most 3 of f and h agree on $f^{-1}(f(R_{n-2}(f)))$. Thus $f^{-1}(f(R_{n-2}(f))) \subset h^{-1}(h(R_{n-2}(h)))$, so that $B_h \subset B_f \cap h^{-1}(h(R_{n-2}(h)))$.

Since $\dim B_h \leq n - 3$ and $B_h \subset h^{-1}(h(R_{n-2}(h)))$, there exists (2.8) k such that, for each $y \in N^n$, $h^{-1}(y)$ has precisely k components. Since $h|f^{-1}(f(R_{n-2}(f))) = f|f^{-1}(f(R_{n-2}(f)))$ and $h(M^n - f^{-1}(f(R_{n-2}(f)))) \subset N^n - f(R_{n-2}(f))$ (by $\mathfrak{P}(a)$ and $\mathfrak{P}(b)$), for each $y \in f(R_{n-2}(f))$, $f^{-1}(y)$ has precisely k components also. For each $y \in N^n - f(R_{n-2}(f))$, there exists i such that $y \in V_i$; there exists J such that $\bar{V}_i \cap \bigcup_{j=J+1}^{\infty} \bar{U}_j = \emptyset$, and thus ($\mathfrak{P}(a)$ and $\mathfrak{P}(b)$) $h^{-1}(\bar{V}_i) = f_J^{-1}(\bar{V}_i)$ and f_J agrees with h on $h^{-1}(\bar{V}_i)$. As a result, $h^{-1}(y)$ has exactly k components. By $\mathfrak{P}(g)$, $f_J^{-1}(y)$ and $f^{-1}(y)$ have the same number of components. Thus f satisfies conclusion (1) of 2.1.

3.6. Proof of Theorem (1.1). First, for the uniqueness, suppose that hg and $\bar{h}\bar{g}$ are two factorizations with intermediate manifolds K^n and L^n . For each point y in K^n , define $\alpha(y) \in L^n$ as the single point $\bar{g}(g^{-1}(y))$. That α is one-to-one and onto is immediate. Since $\bar{h}\alpha = h$, α is locally a diffeomorphism and thus is a diffeomorphism.

For $n = 1$ and 2 , f is, by definition, a local homeomorphism; for $n = 3$, f is a local homeomorphism by [4, p. 469] and [5, p. 91]. Thus f is a k -to-1 C^m covering map [18, p. 128], and the existence of the factoring follows from (2.1). For $n \geq 4$ it follows from (2.1), (3.2) and (3.5).

3.7. REMARKS. The manifold K^n need not be C^m diffeomorphic to M^n . Let $M^n = S^7$, N^n be a C^∞ homotopy 7-sphere other than S^7 , and let f be a homeomorphism of M^n onto N^n which is a C^∞ diffeomorphism except at one point [13]; we may suppose that f itself is C^∞ everywhere (e.g., by the argument of [5, p. 95, (3.3)]). In any factorization K^n is C^∞ diffeomorphic to N^n , and thus is not diffeomorphic to M^n .

In the special case that $f(B_f) \subset f(R_{n-3})$, $\pi_1(N^n) = 0$ and f is C^n , it follows from [6, (1.1)] for $p = n$, $m = 1$, and $k = n - 3$ that $\pi_1(N^n - f(R_{n-3})) = 0$.

Since $\dim(f(R_{n-3})) \leq n - 3$ [5, (1.3)] and $\dim(B_f) \leq n - 3$, $\dim(f^{-1}(f(R_{n-3}))) \leq n - 3$; thus $M^n - f^{-1}(f(R_{n-3}))$ is connected and the restriction map $f| [M^n - f^{-1}(f(R_{n-3}))]$ is a homeomorphism. Hence f is monotone, and (1.1) is immediate in this case.

A map f satisfying the hypothesis of (1.1) is not a branched covering in the sense of Fox [7, p. 250] unless it is actually a covering map.

In case the degree is 0, the conclusion of (1.2) can be improved.

3.8. REMARK. *If M^n and N^n are compact connected oriented C^1 n -manifolds, and $f: M^n \rightarrow N^n$ is C^1 with degree 0, then $\dim(B_f) \geq n - 1$.*

Proof. Suppose the contrary, i.e., that f has degree 0 and $\dim(B_f) \leq n - 2$. Since B_f does not separate M^n , the Jacobian of f is either non-negative or nonpositive (well defined by [21, p. 341, Lemma 3]). By [19, §5] $\dim(f(R_{n-1})) \leq n - 1$, so there exists y with $f^{-1}(y) \subset R_n - R_{n-1}$; let x_k ($k = 1, 2, \dots, m$) be the points of $f^{-1}(y)$. The result follows from [21, Lemma 2 and Theorem 2].

If f is not differentiable but $f(B_f) \neq N^n$, essentially the same proof is valid. The author is grateful to R. F. Williams for suggesting this proof, which is simple than the author's version.

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